Signal and noise transfer in spatiotemporal quantum-based imaging systems

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Fourier-based transfer theory is extended into the temporal domain to describe both spatial and temporal noise processes in quantum-based medical imaging systems. Lag is represented as a temporal scatter in which the release of image quanta is delayed according to a probability density function. Expressions describing transfer of the spatiotemporal Wiener noise power spectrum through quantum gain and scatter processes are derived. Lag introduces noise correlations in the temporal domain in proportion to the correlated noise component only. The effect of lag is therefore dependent on both spatial and temporal physical processes. A simple model of a fluoroscopic system shows that image noise is reduced by a factor that is similar to Wagner’s information bandwidth integral, which depends on the temporal modulation transfer function. © 2007 Optical Society of America

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1. INTRODUCTION

The signal-to-noise ratio (SNR) of an x-ray image can be described in terms of the Fourier-based noise-equivalent number of quanta (NEQ) of the image [1,2]. The NEQ is attractive to use as it expresses the SNR on an absolute scale, independent of image scaling factors. Shaw [2] showed that the NEQ is equal to the square of the image SNR and can be expressed as

\[ \text{NEQ}(k) = \frac{\tilde{q}^2 \tilde{G}^2 \text{MTF}^2(k)}{\text{NPS}(k)}, \quad (1) \]

where \( \tilde{q} \) is the mean number of x-ray quanta in a uniform (Poisson distributed) distribution incident on the detector, \( \tilde{G} \) is the detector gain relating \( \tilde{q} \) to the corresponding average (dark subtracted) detector signal \( \tilde{d} \), MTF\( (k) \) is the spatial frequency variable (typically in cycles/millimeter) and NPS\( (k) \) is the image Wiener noise power spectrum \( (\text{NPS}) \) [1,3,4]. The NPS describes the spectral decomposition of second-moment statistics in terms of spatial frequencies under wide-sense stationary (WSS) conditions [3,4]. An ideal imaging system would generate images with \( \text{NEQ}(k) = \tilde{q} \).

Closely related to the NEQ is the detective quantum efficiency (DQE) [2,5–8]

\[ \text{DQE}(k) = \frac{\text{NEQ}(k)}{\tilde{q}} = \frac{\tilde{q}^2 \tilde{G}^2 \text{MTF}^2(k)}{\text{NPS}(k)}, \quad (2) \]

describing the ability of an imaging system to produce high-SNR images in terms of a detection efficiency. In Eq. (2), \( \text{NEQ}(k) \) describes how many x-ray quanta an image is “worth” (number of quanta required to give same SNR with an ideal imaging system), while \( \tilde{q} \) describes how many quanta were actually used, which is related to patient radiation exposure. Thus, the DQE is loosely interpreted as a “worth/cost” ratio. An ideal imaging system would have unity DQE over all spatial frequencies of interest. In practice, all imaging systems have a DQE less than unity, and a reduced DQE implies either a reduced image SNR or a required increase in radiation exposure to maintain the same SNR.

The DQE can be calculated theoretically if the terms in Eq. (2) can be determined. Many systems can be represented as a cascade of simple processes such as quantum gain or scattering. Quantum gain occurs when one type of image quanta are converted into another. An example is the conversion of x-rays into light in a phosphor. The second physical process is a lossless quantum scatter, in which each quantum is randomly relocated to a new image location. An example is scatter of light in a phosphor. Rabbani et al. [9] and Barrett and co-workers [10,11] showed that propagation of signal and noise through these processes can be expressed analytically in terms of Fourier-based transfer functions. Yao and Cunningham [12] introduced the idea of parallel cascades and described the NPS cross spectral density required to describe more complex systems [13,14]. By cascading expressions for mean signal MTF\( (k) \) and NPS\( (k) \) through each process, a theoretical expression for the DQE is obtained using Eq. (2).

This cascaded-systems approach [15] is valid for quantum-based linear, shift-invariant and WSS imaging
systems where each gain and scatter process is statistically independent of all others. It has been used to understand noise propagation in many types of imaging systems and to generate theoretical models of the DQE that can be used for optimal system design and to establish performance benchmarks [16–29]. As far as we know, until now this approach has not been applicable to any system having a significant temporal dependance, including fluoroscopic imaging systems and other devices having fast read-out rates approaching the characteristic lag times of the detector (such as computed tomography detectors) or other components. For example, lag in the production of light in a fluoroscopic system may result in optical photons generated by a single x-ray photon being distributed between two or more image frames in a video sequence. This introduces statistical correlations in the temporal domain with the consequence that the NPS must be interpreted as a multidimensional PDF that includes the temporal domain, as described by Siewertsen et al. [30]. Lag forms an inherent frame averaging effect that reduces measured noise in the spatial domain and, through Eq. (2), incorrectly inflates measured DQE values. For this reason, the DQE of fluoroscopic systems is not normally measured under fluoroscopic conditions.

Lag may result from the delayed random release of secondary image quanta, such as delayed optical-photon emission in a phosphor or charge trapping and release in a detector. It has been addressed by a number of authors [19,22,31–35] but only using a deterministic approximation that ignores the fact that lag results from random processes acting on individual image quanta. Under certain conditions the deterministic approximation will be valid, but these conditions have not been established.

An approach that includes the statistical effects of lag is to view lag as a “temporal scatter” where each quantum is relocated in time by a random delay described by a one-sided probability distribution function (PDF). In this article, we extend the transfer-theory approach to include lag by generalizing the established expressions of signal and noise transfer into the spatiotemporal domain. These results are then used in a simple theoretical model of a fluoroscopic system to determine the effect of lag on measurements of image noise.

2. THEORY

A theoretical description of signal and noise in the spatiotemporal domain starts with an incident distribution of quanta having a uniform mean in both space and time, represented by the random spatiotemporal distribution of Dirac δ impulses \( \tilde{q}(\mathbf{r}, t) \) (we will use overhead \( \sim \) to identify random variables):

\[
\tilde{q}(\mathbf{r}, t) = \sum_{n=1}^{\tilde{N}} \delta(\mathbf{r} - \tilde{\mathbf{r}}_n) \delta(t - \tilde{t}_n),
\]

where \( \tilde{N} \) is an integer-valued random variable defining the number of quanta in each realization of the process and \( \tilde{\mathbf{r}}_n, \tilde{t}_n \) are random variables indicating quanta coordinates in space and time.

In this section, transfer functions describing propagation of the quantities \( \tilde{q} \) and NPS(\( \mathbf{k} \)) through spatiotemporal quantum gain and scatter processes are developed theoretically. The derivation is similar to a derivation for spatial distributions by Barrett and Myers [36].

A. Spatiotemporal Quantum Gain

A spatiotemporal quantum gain process converts each point in a distribution over space and time to a random output distribution over space and time, such as the conversion of interacting x-rays into optical quanta. There is no scatter associated with this operation. Thus, for an input \( \tilde{q}_{in}(\mathbf{r}, t) \) similar to Eq. (3), the output is

\[
\tilde{q}_{out}(\mathbf{r}, t) = \tilde{N} \sum_{n=1}^{\tilde{N}} \delta(\mathbf{r} - \tilde{\mathbf{r}}_n) \delta(t - \tilde{t}_n),
\]

where \( \tilde{g}_n \) is an integer-valued random variable with mean \( \mu_{\tilde{g}} \) and variance \( \sigma^2_{\tilde{g}} \) describing the quantum gain associated with the \( n \)th input quantum. The mean and autocovariance in the output distribution \( \tilde{q}_{out}(\mathbf{r}, t) \) are given by Eqs. (A6) and (A19) in Appendix A.

**Special Case: WSS Input Distribution.** For the special case of a WSS \( \tilde{q}_{in}(\mathbf{r}, t) \) over space \( A \) and time \( T \), the probability law \( p(\mathbf{r}, t) \) in Eq. (A6) in Appendix A is distributed uniformly in \( A \) and \( T \), giving \( p(\mathbf{r}, t) = 1/TA \), and the mean of \( \tilde{q}_{out}(\mathbf{r}, t) \) takes the form

\[
\tilde{q}_{out} = \frac{\mu_{\tilde{g}}N}{TA} = \mu_{\tilde{g}}\tilde{q}_{in}
\]

in mm\(^{-2}\) s\(^{-1}\), where \( \tilde{q}_{in} \) is the average input spatiotemporal distribution. The autocorrelation is a function of \( \tau_r = \mathbf{r} - \mathbf{r}' \) and \( \tau_t = t - t' \), and Eq. (A19) simplifies to

\[
K_{\tilde{q}_{out}}(\tau_r, \tau_t) = \mu_{\tilde{g}}^2 K_{\tilde{q}_{in}}(\tau_r, \tau_t) + \sigma^2_{\tilde{g}}\tilde{q}_{in} \delta(\tau_r) \delta(\tau_t).
\]

The Wiener noise power spectrum is equal to the Fourier transform of the autocovariance function

\[
\text{NPS}_{\tilde{q}_{out}}(\mathbf{k}, \nu) = \mu_{\tilde{g}}^2 \text{NPS}_{\tilde{q}_{in}}(\mathbf{k}, \nu) + \sigma^2_{\tilde{g}}\tilde{q}_{in}
\]

in mm\(^{-2}\) s\(^{-1}\).

B. Spatiotemporal Quantum Scatter

A spatiotemporal quantum scatter process is a random point process for which each point is randomly relocated to a new location in space and time. Thus, for an input \( \tilde{q}_{in}(\mathbf{r}, t) \) equal to \( \tilde{q}(\mathbf{r}, t) \) in Eq. (3), the output distribution \( \tilde{q}_{out}(\mathbf{r}, t) \) can be written as

\[
\tilde{q}_{out}(\mathbf{r}, t) = \tilde{N} \sum_{n=1}^{\tilde{N}} \delta(\mathbf{r} - \tilde{\mathbf{r}}_n - \tilde{\Delta}_n) \delta(t - \tilde{t}_n - \tilde{\Delta}_n).
\]

The mean and autocovariance in the output distribution \( \tilde{q}_{out}(\mathbf{r}, t) \) are given by Eqs. (B8) and (B25), respectively, in Appendix B.

**Special Case: WSS Input Distribution.** For the special case of a WSS input distribution, the mean output distribution of image quanta, given by Eq. (B8), reduces to \( \tilde{q}_{out} = \tilde{q}_{out}(\mathbf{r}, t) \) equal to
showing that no quanta are lost by the quantum scatter process. As the conditional probability function \( p(r,t | \rho, \lambda) \) in Eq. (B25) (describing scatter from \( r \) to \( p \) in space and \( t \) to \( \lambda \) in time) is shift invariant, the scatter PDF is a function of scatter displacement only: \( p(r - \rho, t - \lambda) \). Applying these conditions to Eq. (B25) and using Eqs. (A19) and (5) yields

\[
K_{\text{out}}(r', t', t') = \bar{q}_{\text{in}} \delta(r - r') \delta(t - t') \int_{\mathbb{R}^3} p(r, t, \lambda) d^2 r d\lambda
\]

\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(r - \rho, t - \lambda) p(r' - \rho', t' - \lambda') \times R_{q_{\text{in}}} (\rho - \rho', \lambda - \lambda') d^2 \rho d^2 \lambda d\lambda' d\rho' d\lambda' d\lambda''
\]

\[
- \bar{q}_{\text{in}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(r - \rho, t - \lambda) p(r' - \rho, t' - \lambda) d^2 \rho d\lambda
\]

\[
- \bar{q}_{\text{in}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(r, t - \lambda) d^2 \rho d\lambda
\]

\[
\times \bar{q}_{\text{in}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(r' - \rho', t' - \lambda') d^2 \rho' d\lambda'.
\]  

(10)

We define new variables \( (\xi, \zeta) = (r - p, t - \lambda) \) and similarly for \( (\xi', \zeta') \). Since \( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(r - \rho, t - \lambda) d^2 \rho d\lambda = 1 \):

\[
K_{\text{out}}(r' - \rho', t' - \lambda) = \bar{q}_{\text{in}} \delta(r - r') \delta(t - t')
\]

\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(\xi, \zeta) p(\xi', \zeta') \times K_{q_{\text{in}}} (r - r' - (\xi - \xi'), (t - t') - (\xi - \zeta'))
\]

\[
\times d^2 \xi d^2 \zeta d\xi' d\zeta'
\]

\[
- \bar{q}_{\text{in}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p(\xi, \zeta) p((r' - r)
\]

\[
+ \xi, (t' - t) + \zeta) d^2 \xi d\zeta
\]

\[
= \bar{q}_{\text{in}} \delta(t' - t) d^2 \xi d\zeta.
\]  

(11)

Letting \( * \) and \( \ast \) denote convolution and correlation operators, respectively, the autocovariance becomes

\[
K_{q_{\text{out}}} (\tau_r, \tau_t) = [K_{q_{\text{in}}} (\tau_r, \tau_t) - \bar{q}_{\text{in}}] p(\tau_r, \tau_t) \ast \bar{q}_{\text{in}} \delta(\tau_r) \delta(\tau_t).
\]  

(12)

Taking the Fourier transform gives the noise power spectrum for spatiotemporal scatter:

\[
\text{NPS}_{\text{out}}(k, \nu) = [\text{NPS}_{\text{in}}(k, \nu) - \bar{q}_{\text{in}}] |T(k, \nu)|^2 + \bar{q}_{\text{in}} \]  

(13)

in \( \text{mm}^2\text{s}^{-1} \), where \( T(k, \nu) \) is the Fourier transform of \( p(\tau_r, \tau_t) \).

3. RESULTS

A. Spatiotemporal Cascaded Systems

Equations (5), (7), (9), and (13), summarized in Table 1, are the new results required to enable use of a spatiotemporal cascaded-systems approach. They show that spatiotemporal gain and scatter are simple generalizations of the established relations [9] by replacing the spatial mean and scatter transfer function with a spatiotemporal mean and transfer function. Generalization to additional dimensions, such as three-dimensional spatial scatter plus temporal scatter, is similarly straightforward.

Similar to the (spatial) NPS, the spatiotemporal NPS consists of both correlated, [\( \text{NPS}_{\text{in}}(k, \nu) - \bar{q}_{\text{in}}] |T(k, \nu)|^2 \), and uncorrelated, \( \bar{q}_{\text{in}} \), components. Equation (13) shows that lag increases noise correlations in the temporal-frequency domain, but only if the correlated component is nonzero. That is, temporal correlations will always be introduced unless the input quanta are Poisson distributed in all dimensions, where \( \text{NPS}_{\text{in}}(k, \nu) = \bar{q}_{\text{in}} \).

In Eq. (13), there is no requirement that \( T(k, \nu) \) be separable into individual spatial and temporal domains. However, physical processes causing spatial and temporal scatter are often independent. For example, generation of optical photons in a phosphor may be delayed depending on the lifetime of excited states, after which each photon is spatially scattered due to optical transport considerations. If these temporal and spatial mechanisms are independent, then \( T(k, \nu) \) will be separable as \( T(k, \nu) = T_r(k) T_t(\nu) \), and it is easy to show that a spatiotemporal scatter is equivalent to a spatial scatter followed immediately by a temporal scatter, as illustrated in Fig. 1, or vice versa. Spatial and temporal scatter will always commute if there are no intermediate processes.

B. Image Noise in Fluoroscopic X-Ray Imaging Systems

Figure 2 shows a simple cascaded model of a fluoroscopic system consisting of a phosphor optically coupled to an active matrix sensor. Although many considerations are not

<table>
<thead>
<tr>
<th>Description</th>
<th>( \bar{q}_{\text{out}} )</th>
<th>( \text{NPS}_{\text{out}}(k, \nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain</td>
<td>( \mu_{\text{out}} \bar{q}_{\text{in}} )</td>
<td>( \mu_{\text{NPS}}^{\text{out}}(k, \nu) + \sigma_{\text{in}}^2 \bar{q}_{\text{in}} )</td>
</tr>
<tr>
<td>Scatter</td>
<td>( \bar{q}_{\text{in}} )</td>
<td>(</td>
</tr>
</tbody>
</table>
in the model, such as the use of a broad x-ray energy spectrum, the model is useful for describing the effect of lag on image noise.

Our model has a quantum selection process at stage 1 to randomly select x-rays that interact in the phosphor. This is a special case of quantum gain, Eq. (7), in which the mean gain (the quantum efficiency) and variance are \( \alpha \) and \( \alpha(1-\alpha) \) respectively [15]. Each interacting x-ray produces many optical quanta, represented as a quantum gain (stage 2) having mean \( \mu_x \) and variance \( \sigma_x^2 \). We assume that lag in the phosphor delays production of each optical photon randomly according to the PDF \( p_a(\tau) \) with associated temporal MTF \( |T_t(\tau)| \). Each photon is subsequently scattered spatially before leaving the phosphor with the PDF \( p_r(\tau) \) and associated MTF \( |T_r(k)| \). These two effects are represented in stage 3 as a single separable spatiotemporal scatter. Optical coupling of these photons to the optical detector is represented as a quantum selection process in stage 4 with coupling efficiency \( \eta \) and variance \( \eta(1-\eta) \). It is assumed that each interacting photon produces a single electron-hole (e-h) pair. The detector produces a signal proportional to the total number of e-h pairs generated in each element of size \( a_x \times a_y \) over an integration time \( a_t \), represented in stage 5 as a convolution with a rectangular spatiotemporal function given by [15]

\[
\Phi(x, y, t) = \begin{cases} \frac{1}{a_x a_y a_t} & \frac{x}{a_x} \leq \frac{y}{a_y} \leq \frac{x}{a_x} \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases}
\]

The corresponding separable Fourier transform is \( T(k, \nu) = a_x \text{sinc}(\pi a_x u) a_y \text{sinc}(\pi a_y v) a_t \text{sinc}(\pi a_t \nu) e^{i a_t \nu} \), where \( u \) and \( v \) are x and y components of \( k \). This last stage is associated with a deterministic electronic gain stage having mean \( \kappa \). The output signal from stage 5 is the detector presampling signal, a function that, when evaluated at the center position associated with each detector element and at the time point corresponding to the end of integration for each video frame, gives the output digital signal. The output NPS for stage 5 is the NPS of the presampling detector signal and is equal to the actual image NPS with the exception of noise aliasing contributions. A summary of mean signal and corresponding NPS at each stage is shown in Table 2.

1. Presampling NPS of a Single Frame
The first step in describing noise in fluoroscopic systems is to examine noise in a single video frame during a continuous fluoroscopic exposure. For convenience, we look at a frame taken at \( t = 0 \). The presampling NPS (psNPS) of the fluoroscopic frame, \( \text{psNPS}_f(k) \), is derived by integration of the NPS from stage 5 over all temporal frequencies [1,30]:

\[
\text{psNPS}_f(k) = \left[ (\mu_x^2 + \sigma_x^2 - \mu_x) \kappa^2 \eta^2 a_u a_r |T_r(k)|^2 a_r^2 \right. \\
\left. \times \int |T_t(v)|^2 \text{sinc}^2(\pi a_t \nu) d\nu \right]
\]

\[
+ \kappa \eta \mu_x a_u a_r q_u |a_r^2 \text{sinc}^2(\pi a_u u) a_r^2 \text{sinc}^2(\pi a_r v)|,
\]

where \( u \) and \( v \) are x and y components of \( k \). We define

\[
\beta = a_t \int |T_t(v)|^2 \text{sinc}^2(\pi a_t \nu) d\nu = \frac{a_t}{a_L}.
\]

where \( a_L \) is the inverse of the integral of our temporal MTF over all frequencies, equal to Schade’s noise-equivalent passband and similar to the information bandwidth integral with \( |T_t(v)|^2 \) acting as the apodization function described by Wagner and colleagues [37,38]. Since \( a_L \) is always greater than or equal to \( a_t \), the ratio \( \beta \) has values \( 0 \leq \beta \leq 1 \).

Equation (15) is rewritten as

\[
\text{psNPS}_f(k) = \left[ (\mu_x^2 + \sigma_x^2 - \mu_x) \kappa^2 \eta^2 a_u a_r |T_r(k)|^2 \beta \right. \\
\left. + \kappa \eta \mu_x a_u a_r q_u |a_r^2 \text{sinc}^2(\pi a_u u) a_r^2 \text{sinc}^2(\pi a_r v)| \right]
\]

\[
= \beta \text{psNPS}_C(k) |T_r(k)|^2 + \text{psNPS}_U(k),
\]

where \( \text{psNPS}_C \) and \( \text{psNPS}_U \) represent the correlated and uncorrelated components of the psNPS, respectively. It should be noted that once noise aliasing is considered, “folding” of the shifted aliases about the sampling frequency always results in the uncorrelated component being independent of frequency and equal to \( \kappa^2 \eta \mu_x a_u a_r a_r q_u [39] \). The effect of noise aliasing on the correlated component depends on \( T_r(k) \). Although only the correlated com-

\[ q_0(r, t) \rightarrow \quad \text{Selection efficiency} \]
\[ \text{Conversion to light quanta} \quad \text{Spatialtemporal scatter of light quanta} \quad \text{Optical coupling efficiency} \quad \text{Integration in detector element} \]
\[ d(r, t) \]
component is scaled by $\beta$, it is desirable that $\text{psNPS}_C \gg \text{psNPS}_U$ to maximize the DQE, and hence lag will normally decrease image noise.

2. psNPS of a Radiographic Image

The NPS of a radiographic image corresponds to the special case of Eq. (17) where $a_i$ is much larger than the correlation time due to system lag, resulting in $a_L = a_i$ and $\beta = 1$. Thus, we can define the psNPS of a radiographic frame as $\text{psNPS}_R$, where

$$\text{psNPS}_R(k) = \text{psNPS}_C(k)|T_r(k)|^2 + \text{psNPS}_U(k).$$

This equation does not have any temporal-frequency dependence. The product of $a_i q_o$ describes the input spatial distribution of incident quanta.

3. Comparison of the psNPS of a Fluoroscopic Frame and Radiographic Image

The ratio of the fluoroscopic and radiographic presampling NPS is given by the ratio of Eqs. (17) and (18):

$$\frac{\text{psNPS}_F(k)}{\text{psNPS}_R(k)} = \frac{\beta \text{psNPS}_C(k)|T_r(k)|^2 + \text{psNPS}_U(k)}{\text{psNPS}_C(k)|T_r(k)|^2 + \text{psNPS}_U(k)}.$$

(19)

When the correlated component of the psNPS is much larger than the uncorrelated component ($\text{psNPS}_C \gg \text{psNPS}_U$), the ratio is approximately $\beta$ and system lag scales noise in fluoroscopic frames by this factor. With very little lag, $\beta = 1$ and $\text{psNPS}_F(k) = \text{psNPS}_R(k)$ as expected. With more lag, the correlated component is reduced while the uncorrelated component is unchanged.

It is convenient to describe the effects of temporal scatter by defining a normalized psNPS, $\text{psNPS}^N_F$, based on Eq. (17):

$$\text{psNPS}^N_F(k) = \frac{\beta \text{psNPS}_C(k)|T_r(k)|^2 + \text{psNPS}_U(k)}{\text{psNPS}_C(k)|T_r(k)|^2 + \text{psNPS}_U(k)}.$$

(20)

where we have assumed near-Poisson conversion gain such that $(\mu^2_o + \sigma_o^2 - \mu_o) = \mu_o^2$, normally a very good approximation. Figure 3 illustrates the effects of temporal scatter on the normalized psNPS of a hypothetical detector having unity quantum efficiency ($\eta = 1$), Gaussian point-spread function (FWHM $= 1.3$ mm) describing spatial scatter of optical quanta, and both a (realistic) high conversion gain $\mu_o = 200$ and a modest conversion gain $\mu_o = 5$. The line representing $\beta = 1.0$ describes the psNPS of both a radiographic image and a fluoroscopic frame when there is little lag, and $\beta = 0$ corresponds to uncorrelated noise. The high conversion gain plots are representative of systems in which the correlated component of the psNPS is much larger than the uncorrelated component, which is normally desirable. Moderate conversion gain may result from an inadequate number of secondary optical quanta [40] where the uncorrelated component is no longer negligible, resulting in an increase in image noise.

4. DISCUSSION

Use of the Fourier-based spatiotemporal cascaded-systems approach to describe image noise relies on a number of assumptions. All gain and scatter processes must be shift invariant in both space and time. All processes must also be linear, an assumption that is not always well satisfied by lag mechanisms in some phosphors where light production build-up and decay times may differ. These are never completely satisfied, and Fourier-based approaches must be interpreted as approximations.

The fluoroscopic results presented here were derived assuming continuous fluoroscopic exposures. Many modern systems pulse the x-ray beam to introduce a “strobe” effect that improves visualization of moving structures. While not addressed in this article, pulsed fluoroscopy could be described as a temporal cyclostationary [41] input.

The topic of spatiotemporal system modeling is complex and involves many other issues not discussed. For example, visualization of moving structures is influenced by lag, which is not addressed here. Rather, we have described an approach that is a simple generalization to es-

<table>
<thead>
<tr>
<th>Process</th>
<th>Mean Signal$^a$</th>
<th>NPS$(k, \nu)^b$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>Incident x-rays</td>
<td>$\hat{q}_o$</td>
</tr>
<tr>
<td>1</td>
<td>Selection of interacting x-rays (probability $a$, variance $a(1-a)$)</td>
<td>$a\hat{q}_o$</td>
</tr>
<tr>
<td>2</td>
<td>Conversion (mean gain $\mu_o$, variance $\sigma_o^2$) to optical quanta in phosphor</td>
<td>$\mu_o a\hat{q}_o$</td>
</tr>
<tr>
<td>3</td>
<td>Spatiotemporal scatter (including lag) of optical quanta in phosphor (transfer function $T(k, \nu)$)</td>
<td>$\mu_o a\hat{q}_o$</td>
</tr>
<tr>
<td>4</td>
<td>Optical coupling and conversion to e-h pairs in sensor (probability $\eta$, variance $\eta(1-\eta)$)</td>
<td>$\eta \mu_o a\hat{q}_o$</td>
</tr>
<tr>
<td>5</td>
<td>Integration of e-h pairs in elements and video frames of optical sensor, a convolution with $I(r/a_x, r/a_y, i/a_I + 1/2)$, and electronic gain $\kappa$</td>
<td>$\kappa \eta \mu_o a\hat{q}_o a_x a_y</td>
</tr>
</tbody>
</table>

$^a$The mean signal describes distributions of quanta (mm$^{-2}$ s$^{-1}$) for stages 1 to 4 and a unitless detector signal in stage 5.

$^b$The NPS have units of mm$^{-2}$ s$^{-1}$ for stages 1 to 4 and mm$^2$ s$^{-1}$ in stage 5.
established methods of signal and noise modeling. This temptst one to assume that other metrics, such as the NEQ and DQE, are directly applicable, but this has not been proven. The NEQ describes how well certain structures can be identified in noisy images under certain conditions. The link between a spatiotemporal NEQ and detectability in fluoroscopy, such as investigated by Wilson and colleagues, [42–49] requires additional attention.

5. CONCLUSION

A generalization of Fourier-based noise transfer theory into the spatiotemporal domain has been described for quantum-based imaging systems, laying the theoretical framework required to apply cascaded-systems theory to spatiotemporal systems. Within the constraints of Fourier-based analyses, it is concluded that

1. System lag, in which the release of image quanta are randomly delayed according to a probability density function, can be described as a “temporal scatter.”

2. The propagation of spatiotemporal noise through quantum gain, spatial scatter, and temporal scatter processes in a cascaded model have been derived.

3. The cascading of temporal and spatial scatter commute as long as there are no intermediate processes.

4. Lag reduces the correlated component of the Wiener noise power spectrum by a factor that is similar to Wagner’s information bandwidth integral but has no effect on the uncorrelated component. Thus, the effect of lag depends on both spatial and temporal noise processes.

5. Lag reduces noise in fluoroscopic images as expected but has no effect on radiographic images as long as image acquisition time is large compared with temporal correlation times.

It is shown that both measurements and theoretical models of the DQE must take these concepts into consideration. While we have applied these ideas to radiographic and fluoroscopic imaging, applications of this work will include other high-speed acquisition systems, such as conventional computed tomography (CT) and newer \( \mu \)CT systems.

APPENDIX A: STATISTICS OF SPATIOTEMPORAL QUANTUM GAIN

1. Mean

The expected value of \( \bar{q}_{\text{out}}(r,t) \) defined in Eq. (4) is computed by averaging over \( \{F_{j},T_{j}\} \), where \( 1 \leq j \leq N \). Denoting \( p_{ij}(r_{j},t_{j}) \) as the joint probability law gives

\[
\langle \bar{q}_{\text{out}}(r,t) \rangle = \frac{\int_{T} \cdots \int_{T} \cdots \int_{A} \int_{A} q_{\text{out}}(r,t) p_{ij}(r_{j},t_{j}) \cdots d^{2}r_{j} \cdots d^{2}t_{j} \cdots}{\int_{T} \cdots \int_{T} \cdots \int_{A} \cdots d^{2}r_{j} \cdots d^{2}t_{j} \cdots} = \langle d^{2}r_{j} \rangle \langle d^{2}t_{j} \rangle \langle \bar{q}_{\text{out}} \rangle,
\]

(A1)

where \( (A,T) \) denotes the spatiotemporal space. Combining this result with Eq. (4), and rearranging the order of integrals and summation, gives

\[
\langle \bar{q}_{\text{out}}(r,t) \rangle = \sum_{n=1}^{N} \int_{T} \cdots \int_{T} \cdots \int_{A} \cdots \int_{A} g_{\text{out}}(r-r_{n}) \delta(t-t_{n}) p_{ij}(r_{j},t_{j}) \cdots d^{2}r_{j} \cdots d^{2}t_{j} \cdots = \sum_{n=1}^{N} \langle \bar{q}_{\text{out}} \rangle_{j \neq n}.
\]

(A2)

Since \( \delta(r-r_{n}) \delta(t-t_{n}) \) is independent of \( r_{j} \) and \( t_{j} \) for \( j \neq n \), we can rearrange, giving
\[ \langle \tilde{q}_{\text{out}}(\mathbf{r}, t) \rangle = \sum_{n=1}^{N} \int_{\mathbf{T}} \int_{\mathbf{A}} g_n \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t_n) \times \left( \int_{\mathbf{T}} \cdots \int_{\mathbf{A}} \cdots \int_{\mathbf{A}} p_{ij}(\langle \mathbf{r}_j, t_j \rangle) \langle d^2 \mathbf{r}_j, 1 \leq j \leq N, j \neq n \rangle \langle dt_j, 1 \leq j \leq N, j \neq n \rangle \right) d^2 \mathbf{r}_n dt_n. \]  

(A3)

The expression in large parentheses is the marginal probability of the random variable \((\mathbf{r}_n, t_n)\), and therefore

\[ \langle \tilde{q}_{\text{out}}(\mathbf{r}, t) \rangle = \sum_{n=1}^{N} \int_{\mathbf{T}} \int_{\mathbf{A}} g_n \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t_n) p_n(\mathbf{r}_n, t_n) d^2 \mathbf{r}_n dt_n. \]  

(A4)

It is straightforward to show that the mean input distribution is given by

\[ \langle \tilde{q}_{\text{in}}(\mathbf{r}, t) \rangle = \sum_{n=1}^{N} p_n(\mathbf{r}, t). \]  

(A5)

As the probability law \(p_n(\mathbf{r}, t)\) is similar for all quantum components, the index \(n\) can be dropped, giving

\[ \langle \tilde{q}_{\text{out}}(\mathbf{r}, t) \rangle = \sum_{n=1}^{N} g_n p(\mathbf{r}, t) = N \mu_g p(\mathbf{r}, t) = \mu_g \langle \tilde{q}_{\text{in}}(\mathbf{r}, t) \rangle. \]  

(A6)

2. Autocorrelation

The spatiotemporal autocorrelation function of \(\tilde{q}_{\text{out}}(\mathbf{r}, t)\) is defined by

\[ R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') = \langle \tilde{q}_{\text{out}}(\mathbf{r}, t) \tilde{q}_{\text{out}}(\mathbf{r}', t') \rangle_{(\mathbf{A}, \mathbf{T})}. \]  

(A7)

Letting \(p_{ij}(\langle \mathbf{r}_j, t_j, t_j' \rangle)\) denote the associated joint probability law, and averaging over the random vector variables \(\{\mathbf{r}_j, \mathbf{r}_j', t_j, t_j'\}\) for \(1 \leq j, j' \leq N\) gives

\[ R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') \]

\[ = \int_{\mathbf{T}} \cdots \int_{\mathbf{A}} \cdots \int_{\mathbf{A}} \left( \sum_{n=1}^{N} g_n \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t_n) \right) \times \left( \sum_{n'=1}^{N} g_{n'} \delta(\mathbf{r}' - \mathbf{r}_{n'}) \delta(t' - t_{n'}) \right) p_{ij}(\langle \mathbf{r}_j, \mathbf{r}_j', t_j, t_j' \rangle) \times \langle d^2 \mathbf{r}_j, 1 \leq j \leq N \rangle \langle d^2 \mathbf{r}_{j'}, 1 \leq j' \leq N \rangle \langle dt_j, 1 \leq j \leq N \rangle \langle dt_{j'}, 1 \leq j' \leq N \rangle \]  

(A8)

Reordering integrals and summations and moving integral elements associated with \(j \neq n\) and \(j' \neq n'\) gives

\[ R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') \]

\[ = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_{\mathbf{T}} \int_{\mathbf{A}} \int_{\mathbf{A}} g_n g_{n'} \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{r}' - \mathbf{r}_{n'}) \delta(t - t_n) \delta(t' - t_{n'}) \times \langle d^2 \mathbf{r}_n, 1 \leq j \neq n \rangle \langle d^2 \mathbf{r}_{n'}, 1 \leq j' \leq N, j' \neq n' \rangle \langle dt_n, 1 \leq j \leq N, j \neq n \rangle \]  

(A9)

The expression in parentheses is the marginal probability of \((\mathbf{r}_n, \mathbf{r}_{n'}, t_n, t_{n'})\), giving

\[ R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_{\mathbf{T}} \int_{\mathbf{A}} \int_{\mathbf{A}} g_n g_{n'} \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{r}' - \mathbf{r}_{n'}) \delta(t - t_n) \delta(t' - t_{n'}) \times \langle d^2 \mathbf{r}_n d^2 \mathbf{r}_{n'} dt_n dt_{n'}. \]  

(A10)

This result is simplified separately for \(n' = n\) and \(n' \neq n\). For \(n' = n\):

\[ R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t')_{|n' = n} = \sum_{n=1}^{N} \int_{\mathbf{T}} \int_{\mathbf{A}} g_n^2 \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{r}' - \mathbf{r}_n) \delta(t - t_n) \delta(t' - t_n) \times \langle d^2 \mathbf{r}_n dt_n. \]  

(A11)
The equation in large parentheses is the marginal probability of $(\mathbf{r}_n, \tilde{\mathbf{z}}_n)$, and the integration reduces it to $p_n(\mathbf{r}_n, t_n)$. The index $n$ in $(\mathbf{r}_n, t_n)$ appears only as an integration variable, and hence if we make the temporary substitutions $(\rho, \lambda) = (\mathbf{r}_n, t_n)$ and note that $\delta(\mathbf{r} - \mathbf{r}_n)\delta(\mathbf{r}' - \mathbf{r}_n) = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{r} - \mathbf{r}_n)$ and similarly in $t$, we obtain

$$R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t')|_{n' = n} = \sum_{n=1}^{N} \int_T \mathcal{A} \left[ \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t') \delta(t - \lambda) \right] \frac{\partial p_n(\rho, \lambda)}{\partial \rho} \frac{\partial^2 p_n(\rho, \lambda)}{\partial \lambda^2} \delta(\mathbf{r}, \mathbf{r}_n) \delta(t - t_n)$$

$$= \sum_{n=1}^{N} \int_T \mathcal{A} \left[ \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t') \right] p_n(\mathbf{r}, t). \quad (A12)$$

Similarly for the case of $n' \neq n$ in Eq. (A10), we obtain

$$R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t')|_{n' \neq n} = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_T \mathcal{A} \left[ \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t_n) \right] \times \left( \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}') \delta(t - \lambda') \right) p_{n,n'}(\mathbf{r}_n, \mathbf{r}', t, t_n) d^2 \mathbf{r}_n d\lambda_n$$

$$= \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_T \mathcal{A} \left[ \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}_n) \delta(t - t_n) \right] p_{n,n'}(\mathbf{r}_n, \mathbf{r}', t, t_n) d^2 \mathbf{r}_n d\lambda_n$$

$$= \sum_{n=1}^{N} \sum_{n'=1}^{N} g_{n,n'}(\mathbf{r}, \mathbf{r}', t, t'). \quad (A13)$$

Combining Eqs. (A12) and (A13) gives the autocorrelation of $\tilde{q}_{\text{out}}(\mathbf{r}, t)$ as

$$R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=1}^{N} \int_T \mathcal{A} \left[ \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \right] p_n(\mathbf{r}, t) + \sum_{n=1}^{N} \sum_{n'=1}^{N} g_{n,n'}(\mathbf{r}, \mathbf{r}', t, t'). \quad (A14)$$

It can also be easily shown from this result that the input autocorrelation function is given by summing

$$R_{\text{in}}(\mathbf{r}, \mathbf{r}', t, t')|_{n' = n} = \sum_{n=1}^{N} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') p_n(\mathbf{r}, t) \quad \text{and} \quad R_{\text{in}}(\mathbf{r}, \mathbf{r}', t, t')|_{n' \neq n} = \sum_{n=1}^{N} \sum_{n'=1}^{N} p_{n,n'}(\mathbf{r}_n, \mathbf{r}', t, t'). \quad (A15)$$

We now make three approximations. The first is to assume that $p_{n,n'}(\mathbf{r}, \mathbf{r}', t, t')$ is similarly distributed for all quanta, which is true if there are no interactions between quanta. The second is to note that there are always a large number of quanta forming an image, such that $N \gg 1$. The third is to assume $N \gg \sigma^2 + \mu^2$ which is also normally a very good approximation. Since, $\Sigma_{n=1}^{N} \delta \tilde{q}^2 = N(\tilde{q}^2) = N(\sigma^2 + \mu^2)$, the autocorrelation of $\tilde{q}_{\text{out}}(\mathbf{r}, t)$ is then given as

$$R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') = \sum_{n=1}^{N} \int_T \mathcal{A} \left[ \int_{\mathcal{T}} \delta^2 \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \right] p_n(\mathbf{r}, t) + \sum_{n=1}^{N} \sum_{n'=1}^{N} g_{n,n'}(\mathbf{r}, \mathbf{r}', t, t'). \quad (A16)$$

A similar derivation shows that the autocorrelation of the input distribution $\tilde{q}_{\text{in}}(\mathbf{r}, t)$ is given by

$$R_{\text{in}}(\mathbf{r}, \mathbf{r}', t, t') = R_{\text{in}}(\mathbf{r}, \mathbf{r}', t, t')|_{n' = n} + R_{\text{in}}(\mathbf{r}, \mathbf{r}', t, t')|_{n' \neq n}$$

$$= N p(\mathbf{r}, t) \delta(\tilde{q} - \tilde{q}') \delta(t - t') \quad + N^2 p(\mathbf{r}, \mathbf{r}', t, t'). \quad (A17)$$

Thus, the output autocorrelation after a quantum gain process can be written in terms of the input autocorrelation as

$$R_{\text{out}}(\mathbf{r}, \mathbf{r}', t, t') = \mu^2 R_{\text{in}}(\mathbf{r}, \mathbf{r}', t, t') \quad + N \sigma^2 p(\mathbf{r}, t) \delta(\tilde{q} - \tilde{q}') \delta(t - t'). \quad (A18)$$
3. Autocovariance
The autocovariance of $\tilde{q}_{\text{out}}(r,t)$ is given by

$$K_{\text{out}}(r,r',t,t') = R_{\text{out}}(r,r',t,t') - \langle \tilde{q}_{\text{out}}(r,t) \rangle \langle \tilde{q}_{\text{out}}(r',t') \rangle$$

$$= \frac{\mu^2}{\sigma^2} \delta(r - r') \delta(t - t') - \frac{N^2 \sigma^2}{\sigma^2} \delta(r - r') \delta(t - t') - \frac{N^2}{\sigma^2} \mu^2 \delta(r - r') \delta(t - t').$$

\(\text{(A19)}\)

APPENDIX B: STATISTICS OF SPATIOTEMPORAL QUANTUM SCATTER

1. Mean
The expected value of $\tilde{q}_{\text{out}}(r,t)$ after a scatter operation defined in Eq. (8) is computed by taking an average over $\{r_j,t_j\}$ for $1 \leq j \leq N$ and an average over the random variables $\{\Delta r_j, \Delta t_j\}$ representing the scatter displacement vectors:

$$\langle \tilde{q}_{\text{out}}(r,t) \rangle = \mathbb{E}_{\{r_j,t_j\}} \mathbb{E}_{\{\Delta r_j,\Delta t_j\}} \langle \tilde{q}_{\text{out}}(r,t) \rangle,$$

\(\text{(B1)}\)

where $\mathbb{E}{}$ indicates the expectation operation. The joint probability law of displacement vectors given the initial locations $\{r_j,t_j\}$ is given by $p_{\{j\}}(r_j + \Delta r_j, t_j + \Delta t_j | r_j, t_j)$. The first step is to average over the possible displacements, giving $\tilde{q}_{A}(r,t)$ as

$$\tilde{q}_{A}(r,t) = \mathbb{E}_{\{\Delta r_j,\Delta t_j\}} \langle q_{\text{out}}(r,t) \rangle$$

$$= \int_T \cdots \int_T \int_A \cdots \int_A \left( \sum_{n=1}^{N} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \right)$$

$$\times p_{\{j\}}(r_j + \Delta r_j, t_j + \Delta t_j | r_j, t_j) \langle \tilde{q}_{\text{out}}(r,t) \rangle$$

$$\times \{d\Delta r_j, 1 \leq j \leq N, j \neq n\} \langle \tilde{q}_{\text{out}}(r,t) \rangle.$$

\(\text{(B2)}\)

Changing the order of sumations and integrals, and noting that the $\delta$ functions appear in integrals for $j = n$, yields

$$\tilde{q}_{A}(r,t) = \sum_{n=1}^{N} \int_T \cdots \int_T \int_A \cdots \int_A \left( \sum_{j=n}^{N} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \right)$$

$$\times p_{\{j\}}(r_j + \Delta r_j, t_j + \Delta t_j | r_j, t_j) \langle \tilde{q}_{\text{out}}(r,t) \rangle$$

$$\times \{d\Delta r_j, 1 \leq j \leq N, j \neq n\} \langle \tilde{q}_{\text{out}}(r,t) \rangle.$$

\(\text{(B3)}\)

The PDF $p_{\{j\}}$ is $3N$-dimensional. The expression in parentheses is the marginal probability and integration over indices $j$ for $j \neq n$ reduces the dimensionality to 3 and $p_{\{\}}$ becomes $p_n$. In addition, we write $(r_\Delta, \lambda_\Delta) = (r_n + \Delta r_n, t_n + \Delta t_n)$, giving

$$\tilde{q}_{A}(r,t) = \sum_{n=1}^{N} \int_T \cdots \int_T \int_A \delta(r - r_\Delta) \delta(t - \lambda_\Delta) p_n(r_n, t_n) \langle \tilde{q}_{\text{out}}(r,t) \rangle$$

$$\times \{d^2 r_n, 1 \leq j \leq N, j \neq n\}.$$

\(\text{(B4)}\)

The next step is to average over the initial locations $\{r_j, t_j\}$. If $p_{\{j\}}(r_j, t_j)$ denotes the corresponding joint probability law, then

$$\langle \tilde{q}_{\text{out}}(r,t) \rangle = \mathbb{E}_{\{r_j,t_j\}} \langle \tilde{q}_{A}(r,t) \rangle$$

$$= \sum_{n=1}^{N} \int_T \cdots \int_T \int_A \int_T \cdots \int_A \int_A \int_T \cdots \int_A \int_A \cdots \int_A \int_A$$

$$\times p_{\{j\}}(r_j, t_j) \langle \tilde{q}_{\text{out}}(r,t) \rangle.$$

\(\text{(B5)}\)

Following steps similar to those used above gives

$$\langle \tilde{q}_{\text{out}}(r,t) \rangle = \sum_{n=1}^{N} \int_T \cdots \int_T \int_A \int_T \cdots \int_A \int_A \cdots \int_A \int_A$$

$$\times p_{\{j\}}(r_j, t_j) \langle \tilde{q}_{\text{out}}(r,t) \rangle.$$

\(\text{(B6)}\)

The quantity in large parentheses is the marginal probability of the random variable $(r_n, t_n)$, giving
\[
\langle \tilde{q}_{\text{out}}(r,t) \rangle = \sum_{n=1}^{N} \int_{T} \int_{A} p_n(r,t) p_n(r',t') d^2r' dt'.
\]  
(B7)

Using Eq. (A5) and noting that \( p_n(r,t|r',t') \) is similar for all quanta gives

\[
\langle \tilde{q}_{\text{out}}(r,t) \rangle = \int_{T} \int_{A} p(r,t|r',t') \langle \tilde{q}_{\text{out}}(r',t') \rangle d^2r' dt'.
\]  
(B8)

1. Autocorrelation

The autocorrelation of \( \tilde{q}_{\text{out}}(r,t) \) following the spatiotemporal quantum scatter is given by

\[
R_{q_{\text{out}}}(r,r',t,t') = E_{\tilde{q}_{\text{out}}(r,t)}[\langle \tilde{q}_{\text{out}}(r,t) \rangle].
\]  
(B9)

We let \( p_{ij}(r_n+\Delta r_j,r'_j+\Delta r_{j'},t_j+t_{j'},t_j+\Delta t_{j'}|r_j,r_j,t_j,t_j') \) be the joint probability law between the corresponding random vector variables, and, similar to the above calculation of the mean scattered distribution, we first average over spatiotemporal displacements:

\[
R_{q_{\text{out}}}(r,r',t,t') = E_{\tilde{q}_{\text{out}}(r,t)}[\langle \tilde{q}_{\text{out}}(r,t) \rangle]
\]

\[
\quad = \int_{T} \int_{A} \cdots \int_{T} \int_{A} \cdots \int_{A} \left( \sum_{n=1}^{N} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \right) \left( \sum_{n'=1}^{N} \delta(r' - r_{n'} - \Delta r_{n'}) \delta(t' - t_{n'} - \Delta t_{n'}) \right)
\]

\[
\times p_{ij}(r_j,\Delta r_j, r_{j'}, \Delta r_{j'}, t_j + \Delta t_j; t_j + \Delta t_j'|r_j, r_j, t_j, t_j')
\]

\[
\times \{d^2 \Delta r_n, 1 \leq j \leq N \} \{d^2 \Delta r_{n'}, 1 \leq j' \leq N \} \{d \Delta t_j, 1 \leq j \leq N \} \{d \Delta t_{j'}, 1 \leq j' \leq N \}.
\]  
(B10)

Moving the summations and noting that the \( \delta \) functions appear in integrals only for the indices \( j = n \) and \( j' = n' \) gives

\[
R_{q_{\text{out}}}(r,r',t,t') = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_{T} \int_{A} \cdots \int_{T} \int_{A} \cdots \int_{A} \left( \sum_{m} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \right) \left( \sum_{m'} \delta(r' - r_{n'} - \Delta r_{n'}) \delta(t' - t_{n'} - \Delta t_{n'}) \right)
\]

\[
\times p_{ij}(r_j, \Delta r_j, r_{j'}, \Delta r_{j'}, t_j + \Delta t_j; t_j + \Delta t_j'|r_j, r_j, t_j, t_j')
\]

\[
\times \{d^2 \Delta r_n, 1 \leq j \leq N, j \neq n \} \{d^2 \Delta r_{n'}, 1 \leq j' \leq N, j' \neq n' \} \{d \Delta t_j, 1 \leq j \leq N \} \{d \Delta t_{j'}, 1 \leq j' \leq N \} \{d^2 \Delta r_{n'}, d^2 \Delta r_n, d \Delta t_n, d \Delta t_{n'} \}.
\]  
(B11)

The quantities in parentheses are the scatter vector marginal probabilities and, similar to the steps following Eq. (B3), gives

\[
R_{q_{\text{out}}}(r,r',t,t') = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_{T} \int_{A} \cdots \int_{T} \int_{A} \left( \sum_{m} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \right) \left( \sum_{m'} \delta(r' - r_{n'} - \Delta r_{n'}) \delta(t' - t_{n'} - \Delta t_{n'}) \right)
\]

\[
\times p_{ij}(r_n + \Delta r_n, r_{n'}, \Delta r_{n'}, t_n + \Delta t_n, t_{n'} + \Delta t_{n'}, r_n, r_{n'}, t_n, t_{n'})
\]

\[
\times d^2 \Delta r_n d^2 \Delta r_{n'} d \Delta t_n d \Delta t_{n'}.
\]  
(B12)

Similar to the above, again using the sifting property of \( \delta \) functions gives

\[
R_{q_{\text{out}}}(r,r',t,t')_{|n=0} = \sum_{n=0}^{N} \int_{T} \int_{A} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \delta(t' - r_{n'} - \Delta r_{n'})
\]

\[
\times \delta(t' - t_n - \Delta t_n) p_n(r_n + \Delta r_n, t_n + \Delta t_n | r_n, t_n) d^2 \Delta r_n d \Delta t_n
\]

\[
= \sum_{n=1}^{N} \delta(r - r') \delta(t - t') p_n(r,t | r_n, t_n).
\]  
(B13)
The average over initial positions is now considered. If \( p \) which can be rearranged as

\[
R_{\Delta}(r, r', t', t'_n) = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_{\mathbb{T}} \int_{\mathbb{A}} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \\
\times \left( \int_{\mathbb{T}} \int_{\mathbb{A}} \delta(r' - r_n - \Delta r_n) \delta(t' - t_n - \Delta t_n) \right) \\
\times P_{(n,n')} (r_n + \Delta r_n, r_n' + \Delta r_n', t_n + \Delta t_n, t_n' + \Delta t_n) | r_n, r_n', t_n, t_n' \right) \\
\times d^2 \Delta r_n d \Delta t_n.
\]  

(B14)

We again use \(( \rho_\Delta, \lambda_\Delta ) = (r_n + \Delta r_n, t_n + \Delta t_n)\) and similarly for \(( \rho_\Delta', \lambda_\Delta' )\), giving

\[
R_{\Delta}(r, r', t, t'_n) = \sum_{n=1}^{N} \sum_{n'=1}^{N} \int_{\mathbb{T}} \int_{\mathbb{A}} \delta(r - r_n - \Delta r_n) \delta(t - t_n - \Delta t_n) \\
\times P_{(n,n')} (r_n + \Delta r_n, \rho_\Delta, t_n + \Delta t_n) | r_n, r_n', t_n, t_n' \right) \\
\times d^2 \Delta r_n d \Delta t_n
\]  

(B15)

Combining the \( n' = n \) and \( n' \neq n \) cases gives

\[
R_{\Delta}(r, r', t, t'_n) = \sum_{n=1}^{N} \delta(r - r') \delta(t - t') P_n(r, t|r_n, t_n) + \sum_{n=1}^{N} \sum_{n'=1}^{N} P_{(n,n')} (r, r', t, t'| r_n, r_n', t_n, t_n').
\]  

(B16)

The average over initial positions is now considered. If \( p_{(r,t)}(r, t) \) denotes the joint probability between the set of random variables \((r_j, t_j)\), then

\[
R_{\Delta}(r, r', t, t'_n) = \mathbb{E}_{(r_{(j)}, t_{(j)})} (R_{\Delta}(r, r', t, t'_n)) \\
= \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \int_{\mathbb{A}} \cdots \int_{\mathbb{A}} R_{\Delta}(r, r', t, t'_n) P_{(r, t)}(r_{(j)}, t_{(j)}, t_{(j)}) | d^2 r_j, 1 \leq j \leq N \} \\
\times | dt_j, 1 \leq j' \leq N \} | dt_{j'}, 1 \leq j \leq N \} | dt_{j'}, 1 \leq j' \leq N \}.
\]  

(B17)

Combining with Eq. (B16) gives

\[
R_{\Delta}(r, r', t, t'_n) = \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \int_{\mathbb{A}} \cdots \int_{\mathbb{A}} \left( \sum_{i=1}^{N} \delta(r - r') \delta(t - t') P_i(r, t|r_n, t_n) \\
+ \sum_{n=1}^{N} \sum_{n'=1}^{N} P_{(i,n')} (r, r', t, t'| r_n, r_n', t_n, t_n') \right) P_{(r, t)}(r_{(j)}, t_{(j)}, t_{(j)}) \\
\times | d^2 r_j, 1 \leq j \leq N \} | d^2 r_{j'}, 1 \leq j' \leq N \} | dt_j, 1 \leq j \leq N \} | dt_{j'}, 1 \leq j' \leq N \},
\]  

which can be rearranged as
\[
\begin{align*}
R_{q_{\text{out}}}(r, r', t, t') &= \sum_{n=1}^{N} \int_{\mathcal{T}} \int_{\mathcal{A}} \delta(r - r') \delta(t - t') p_n(r, t | r_n, t_n) \left( \int_{\mathcal{T}} \cdots \int_{\mathcal{A}} \cdots \int_{\mathcal{A}} \right) \\
&\quad \times p_{(j', j)}(r, r_j, t, t_j) [d^2r, 1 \leq j \leq N, j \neq n] [d^2r, 1 \leq j' \leq N] \\
&\quad \times [dt, 1 \leq j \leq N, j \neq n] [dt, 1 \leq j' \leq N] d^2r_n dt_n \\
&\quad \times \sum_{n=1}^{N} \sum_{n' \neq n}^{N} \int_{\mathcal{T}} \int_{\mathcal{A}} \int_{\mathcal{A}} \int_{\mathcal{A}} p_{(n, n')} (r, r', t, t' | r_{n'}, r_{n'}, t_{n'}, t_{n'}) \left( \int_{\mathcal{T}} \cdots \int_{\mathcal{A}} \cdots \int_{\mathcal{A}} \right) \\
&\quad \times p_{(j, j)}(r, r_j, t, t_j) [d^2r, 1 \leq j \leq N, j \neq n] [d^2r, 1 \leq j' \neq n'] \\
&\quad \times [dt, 1 \leq j \leq N, j \neq n] [dt, 1 \leq j' \neq N, j' \neq n'] d^2r_n d^2r_{n'} d^2r_{n'} dt_n dt_{n'}. 
\end{align*}
\]
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